Math 3236 Statistical Theory

3/14/23

$X_1 \ldots X_N$ are i.i.d Normal

$\mu \quad \sigma^2$

$$\hat{\mu} = \bar{X} = \frac{1}{N} \sum_i X_i$$

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_i (X_i - \bar{X})^2 = \frac{1}{N-1}\left(\sum_i X_i^2 - N\bar{X}^2\right)$$

$$\sum_i (X_i - \bar{X})^2 = \sum_i X_i^2 - 2\sum_i X_i \bar{X} + \sum_i \bar{X}^2$$

$$= \sum_i X_i^2 - 2N\bar{X}^2 + N\bar{X}^2$$

$$= \sum_i X_i^2 - N\bar{X}^2$$

$\hat{\mu} \quad \hat{\sigma}^2$ are independent

$$\hat{\mu} = N\left(\mu, \frac{\sigma^2}{N}\right)$$

$$\frac{\hat{\sigma}^2}{\sigma^2} = \chi^2_{N-1}$$

X_i are $\mathcal{N}(0,1)$

$\bar{X}$ $\qquad X_i - \bar{X} = Y_i$

$$Y_i = \left(1 - \frac{1}{N}\right) X_i + \sum_{j \neq i} \frac{X_j}{N}$$

$$\operatorname{cov}(\bar{X}, Y_i) = \operatorname{cov}(\bar{X}, X_i) - \operatorname{var}(\bar{X})$$

$$= \frac{1}{N} \sum_j \operatorname{cov}(X_i, X_j) - \operatorname{var}(\bar{X}) =$$

$$= \frac{1}{N} \operatorname{var}(X_i) - \frac{1}{N^2} \sum_j \operatorname{var}(X_j) =$$

$$= 0$$

$$D = \begin{pmatrix} \frac{1}{N} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & C & \\ 0 & & & \end{pmatrix}$$

$$C_{ij} = \mathrm{cov}(Y_i, Y_j)$$

$$f(\bar{x}, y) = \frac{1}{Z} e^{-\frac{\bar{x}^2}{2N} - \frac{(y C^{-1} y)}{2}}$$

$$\bar{X} \perp\!\!\!\perp Y_i$$

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

C $n \times n$ cov. matrix

C_1 $n_1 \times n_1$ C_2 $n_2 \times n_2$

$$\underline{X} = (X_1 \ldots X_n)$$

$$\underline{X}^1 = (X_1 \ldots X_{n_1}) \qquad \underline{X}^2 = (X_{n_1+1} \ldots X_n)$$

$$\underline{X}^1 \perp\!\!\!\perp \underline{X}^2$$

$\hat{\mu} \perp\!\!\!\perp Y_i = X_i - \bar{X}$

$\hat{\mu} \perp\!\!\!\perp \sum_i Y_i^2$

Observe that $\sum_i Y_i = 0$

$Y_2 \quad \dots \quad Y_{N-1}$

$\bar{X} = \frac{1}{N} \sum_i X_i$

$\bar{X} \quad X_2 \quad \dots \quad X_N$

$\underline{v}_1 = \left(\frac{1}{N} \quad \dots \quad \frac{1}{N}\right)$

$\underline{v}_2 = (0, 1, 0 \; 0 \; \dots 0)$

$\underline{v}_N = (0, 0 \; \dots \; 0 \; 1)$

$\underline{w}_1 \quad \dots \quad \underline{w}_N$

$\underline{w}_1 = \left(\frac{1}{\sqrt{N}}, \quad \dots \quad \frac{1}{\sqrt{N}} \right)$

$\underline{w}_2$ orthonormal

$\vdots$ and orthogonal to w_1

$\underline{w}_n$

$$\overline{X} = \frac{1}{\sqrt{N}} \left(\underline{w}_1 \cdot \underline{X} \right) = z_1$$

$$z_i = (w_i \cdot \underline{X}) \qquad i \geq 2$$

$$\text{cov}(z_i, z_j) = \sum_{k,l} w_{i,k} \, w_{j,l} \, \text{cov}(X_k, X_l) =$$

$$\underline{w}_i = (w_{i,1} \quad \dots \quad w_{i,N})$$

$$= \sum_k w_{i,k} \, w_{j,k} = \begin{cases} 0 & \text{if } j \\ 1 & i = j \end{cases}$$

$z_2 \quad \dots \quad z_N$ are Normal Standard

independent

$$\sum_{i=2}^{N} z_i^2 = \chi^2_{N-1}$$

$$\sum_{i=1}^{N} Z_i^2 = \sum_i X_i^2$$

$$\sum_{i=2}^{N} Z_i^2 = \sum_{i=1}^{N} X_i^2 - Z_1^2 = \sum_{i=1}^{N} X_i^2 - N\bar{X}^2$$

$$= \hat{\sigma}^2$$

Useful fact: if X_i are ind. $N(0,1)$ and O is an orthonormal matrix

$$Y_i = (O\underline{X})_i = \sum_j O_{ij} X_j$$

Y_i are ind. $N(0,1)$.

X_i are indep. $N(\mu, \sigma^2)$

$Y_i = \frac{X_i - \mu}{\sigma}$ are indep. $N(0,1)$

$X_i = \sigma Y_i + \mu$

$$\frac{1}{N}\sum_i Y_i = \frac{1}{N}\sum_i \frac{(X_i - \mu)}{\sigma}$$

$$\frac{1}{N}\sum_i X_i = \frac{\sigma}{N}\sum_i Y_i + \mu$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{N}\right) = \hat{\mu}$$

$$\sum_i (X_i - \bar{X})^2 = \hat{\sigma}^2$$

$$\hat{\sigma}^2 = \sum_i (\sigma Y_i - \sigma \bar{Y})^2 = \sigma^2 \sum_i (Y_i - \bar{Y})^2$$

$$\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{N-1}$$

$$X_i \quad \text{i.i.d} \quad N(\mu, \sigma^2)$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{N}\right)$$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{N}} \sim N(0,1)$$

$$U = \frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{N-1}$$

σ

$$\hat{\mu} = \bar{X} \qquad \hat{\sigma} = \left(\frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})^2 \right)^{1/2}$$

Find N such That

$$p = \mathbb{P}\left(|\hat{\mu} - \mu| \le \frac{1}{5}\sigma \text{ and } |\hat{\sigma} - \sigma| \le \frac{1}{5}\sigma \right) \geq \frac{1}{2}$$

$$\left| \frac{\hat{\sigma}}{\sigma} - 1 \right| \le \frac{1}{5} \quad \text{Two sided}$$

Different Example

$$\frac{\hat{\sigma}}{\sigma} \geq 0.9 \quad \text{one sided}$$

$$\Downarrow$$

$$\hat{\sigma} \geq 0.9\sigma$$

$$\Downarrow$$

$$\sigma \le \hat{\sigma}/0.9$$

$$p = \underbrace{\mathbb{P}\left(|\hat{\mu} - \mu| < \frac{1}{5}\sigma \right)}_{p_1(N)} \underbrace{\mathbb{P}\left(|\hat{\sigma} - \sigma| < \frac{1}{5}\sigma \right)}_{p_2(N)}$$

$$p_1 = \mathbb{P}\left(\frac{\sqrt{N}|\hat{\mu} - \mu|}{\sigma} \leq \frac{1}{5}\sqrt{N} \right)$$

$$U = \frac{\sqrt{N}(\hat{\mu} - \mu)}{\sigma} \simeq N(0,1)$$

$$p_1 = \mathbb{P}\left(|U| \leq \frac{1}{5}\sqrt{N} \right)$$

$$p_2 = \mathbb{P}\left(0.8 \leq \frac{\hat{\sigma}}{\sigma} \leq 1.2 \right)$$

$$V = N\frac{\hat{\sigma}^2}{\sigma^2} \simeq \chi^2_{N-1}$$

$$p_2 = \mathbb{P}\left(0.64N \leq V \leq 1.44N \right)$$

Find the minimum N such that

$$p_1(N)\,p_2(N) \geq \frac{1}{2}$$

$N = 21$,